Quadratic solitons as nonlocal solitons

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We show that quadratic solitons are equivalent to solitons of a nonlocal Kerr medium. This provides new physical insight into the properties of quadratic solitons, often believed to be equivalent to solitons of an effective saturable Kerr medium. The nonlocal analogy also allows for novel analytical solutions and the prediction of novel bound states of quadratic solitons.

Quadratic nonlinear (or $\chi^{(2)}$) materials have a strong and fast electronic nonlinearity, which makes them excellent materials for the study of nonlinear effects, such as solitons [1]. The main properties of quadratic solitons are well-known [2] and both 1+1 [3] and 2+1 [4] dimensional bright spatial solitons have been observed experimentally. Unlike conventional solitons, which form due to a self-induced refractive index change, the formation of quadratic solitons does not involve any change of the refractive index. Thus the underlying physics of quadratic solitons is often obscured by the mathematical model. Only recently Assanto and Stegeman used the cascading phase shift and parametric gain to give an intuitive interpretation of effects, such as self-focusing, defocusing, and soliton formation in $\chi^{(2)}$ materials [5].

Nevertheless certain features of quadratic solitons, such as soliton interaction, are still without a physical interpretation. Here we use the analogy between parametric interaction and nonlocality and present a physically intuitive nonlocal theory, which is exact in predicting the profiles of stationary quadratic solitons and which provides a simple physical explanation for their properties including formation of bound states. The nonlocal analogy was applied recently by Shadrivov and Zharov to find approximate bright quadratic soliton solutions, but the nonlocal concept was not fully exploited to give a broad physical picture in the whole regime of excistence [6].

We consider a fundamental wave (FW) and its second harmonic (SH) propagating along the z-direction in a quadratic nonlinear medium under conditions for type I phase-matching. The normalized dynamical equations for the slowly varying envelopes $E_{1,2}(x,z)$ are then [7]

$$i\partial_z E_1 + d_1 \partial_x^2 E_1 + E_1^* E_2 \exp(-i\beta z) = 0, \quad (1)$$
$$i\partial_z E_2 + d_2 \partial_x^2 E_2 + E_1^2 \exp(i\beta z) = 0. \quad (2)$$

In the spatial domain $d_1 \approx 2d_2$, $d_j > 0$, and the coordinate x represents a transverse spatial direction. The term $\partial_x^2 E_j$ then represents beam diffraction. In the temporal domain d_j is arbitrary and x represents time. In this case $\partial_x^2 E_j$ represents pulse dispersion. The parameter β

is the normalized phase-mismatch and j=1,2.

Physical insight into the properties of Eqs. (1-2) may be obtained from the cascading limit, in which the phase-mismatch is large, $\beta^{-1} \to 0$. Writing $E_2 = e_2 \exp(i\beta z)$ and assuming slow variation of $e_2(x,z)$ gives the nonlinear Schrödinger (NLS) equation $i\partial_z E_1 + d_1 \partial_x^2 E_1 + \beta^{-1}|E_1|^2 E_1 = 0$, in which the *local* Kerr nonlinearity is due to the coupling to the SH field $e_2 = E_1^2/\beta$. The SH is thus slaved to the FW and the widths of the SH and FW are fixed. The sign of the mismatch β determines whether the effective Kerr nonlinearity is focusing or defocusing and thus the cascading limit predicts that both bright and dark quadratic solitons can exist.

However, even for stationary solutions the NLS equation is inaccurate, since the term $\partial_x^2 E_2$ is neglected. Thus it predicts that in higher dimensions bright solitons are unstable and will either spread out or collapse [8], whereas it is known that stable quadratic solitons exist in all dimensions and that collapse cannot occur in the $\chi^{(2)}$-system (1-2) [9, 10]. The stabilizing effect of the $\chi^{(2)}$ nonlinearity is often described as being due to saturation of the effective Kerr nonlinearity [5, 9, 11]. We show below that the nonlinearity is in fact nonlocal.

To obtain a more accurate model than that given by the cascading limit we assume a slow variation of the SH field $e_2(x,z)$ in the propagation direction only (i.e., only $\partial_z e_2$ is neglected). The SH is still expressed in terms of the FW, but now the relation has the form of a convolution, leading to the *nonlocal equation for the FW*

$$i\partial_z E_1 + d_1 \partial_x^2 E_1 + \beta^{-1} N(E_1^2) E_1^* = 0, \quad (3)$$
$$N(E_1^2) = \int_{-\infty}^{\infty} R(x-\xi) E_1^2(\xi, z) d\xi, \quad (4)$$

with $E_2 = \beta^{-1} N \exp(i\beta z)$. Equations (3-4) clearly show that the interaction between the FW and SH is equivalent to the propagation of a FW in a medium with a nonlocal nonlinearity. In the Fourier domain (denoted with tilde) the response function $R(x)$ is a Lorentzian $\widetilde{R}(k) = 1/(1 + s\sigma^2 k^2)$, where $\sigma = |d_2/\beta|^{1/2}$ represents the

degree of nonlocality and $s=\text{sign}(d_2\beta)$. For $s=+1$, where the $\chi^{(2)}$-system (1-2) has a family of bright (for $d_1>0$) and dark (for $d_1<0$) soliton solutions [12], $\widetilde{R}(k)$ is positive definite and localized, giving

$$R(x) = (2\sigma)^{-1} \exp(-|x|/\sigma). \qquad (5)$$

The cascading limit $\beta^{-1} \to 0$ is now seen to correspond to the local limit $\sigma \to 0$, in which the response function becomes a delta function, $R(x) \to \delta(x)$.

With the nonlocal analogy one can assign simple physically intuitive pictures to several important cases. When the mismatch $|\beta|$ is large Eqs. (3-4) reduce to the so-called weakly nonlocal equation with $\sigma \ll 1$ [13]. Similarly the nearly phase-matched limit when $\beta \approx 0$ corresponds to the strongly nonlocal limit with $\sigma \gg 1$, when Eqs. (3-4) become effectively linear [14, 15].

For $s=-1$, $\widetilde{R}(k)$ has poles on the real axis and the response function becomes oscillatory in nature with the Cauchy principal value $R(x)=(2\sigma)^{-1}\sin(|x|/\sigma)$. In this case the propagation of solitons has a close analogy with the evolution of a particle in a nonlinear oscillatory potential. In fact, it is possible to show that the oscillatory response function explains the fact that dark and bright quadratic solitons radiate linear waves for $s=-1$ [12].

It is important to notice that in contrast to the conventional nonlocal NLS equation describing, e.g., materials with a thermal [17] or diffusion [18] type nonlinearity, liquid crystals [19], and photorefractive crystals [20], the nonlocal response of the $\chi^{(2)}$ system depends on the square of the FW, not its intensity. Thus the phase of the FW enters into the picture and one cannot directly transfer the known nonlocal stability results for plane waves [14, 16] and solitons [13, 15].

However, for stationary fields the nonlocal system (3-4) represents an exact model for $\chi^{(2)}$ materials. Therefore the known properties of nonlocal solitons in terms of profiles directly apply to quadratic solitons. Consider stationary solutions to Eqs. (1-2) in the form

$$E_1(x, z) = a_1\phi_1(\tau) \exp(i\lambda z), \qquad (6)$$

$$E_2(x, z) = a_2\phi_2(\tau) \exp(i2\lambda z + i\beta z), \qquad (7)$$

where the profile $\phi_j(\tau)$ is real, with $\tau = x|\lambda/d_1|^{1/2}$, $a_1^2 = \lambda^2|d_2/(2d_1)|$, and $a_2=\lambda$. This scaling reduces the number of free parameters to one and transforms Eqs. (1-2) into the following system [12]

$$s_1\phi_1'' - \phi_1 + \phi_1\phi_2 = 0, \ s_2\phi_2'' - \alpha\phi_2 + \phi_1^2/2 = 0, \qquad (8)$$

where $s_j=\text{sign}(\lambda d_j)=\pm 1$, $\alpha=(2+\beta/\lambda)|d_1/d_2|$, and prime denotes differentiation with respect to the argument. The properties of solitons described by Eqs. (8) are well-known [12]. A family of bright (dark) solitons exist for $s_2=s_1=+1$ ($s_2=-s_1=+1$) and $\alpha>0$. As discussed above we do not consider the combinations $s_2=\pm s_1=-1$, for which solitons do not exist in the whole α-space.

The second equation in Eqs. (8) has the formal solution $\phi_2=\gamma N(\phi_1^2)$, with $\gamma=1/(2\alpha)$ and the nonlocal nonlinearity $N(\phi_1^2)$ given by Eq. (4). For $\text{sign}(s_2\alpha)=+1$ the

response function is $R(\tau)=(2\bar{\sigma})^{-1}\exp(-|\tau|/\bar{\sigma})$, with the degree of nonlocality $\bar{\sigma}=|\alpha|^{-1/2}$. Inserting the SH into the first equation in Eqs. (8) then gives the exact nonlocal model for the FW in the $\chi^{(2)}$ system (8)

$$s_1\partial_\tau^2\phi_1 - \phi_1 + \gamma\phi_1 \int_{-\infty}^{\infty} R(\tau - \xi)\phi_1^2(\xi)d\xi = 0, \qquad (9)$$

where γ is the strength of the nonlocal nonlinearity. Thus $\chi^{(2)}$ solitons are equivalent to nonlocal solitons.

In the weakly nonlocal case $\bar{\sigma}\ll 1$ (i.e., $|\alpha|\gg 1$) the response function $R(\tau)$ is much narrower than the FW intensity ϕ_1^2. Taylor expanding ϕ_1^2 under the integral in Eq. (9) we then obtain the weakly nonlocal model [13]

$$s_1\partial_\tau^2\phi_1 - \phi_1 + \gamma(\phi_1^2 + \bar{\sigma}^2\partial_\tau^2\phi_1^2)\phi_1 = 0, \qquad (10)$$

where $\phi_2=\gamma(1 + \bar{\sigma}^2\partial_\tau^2)\phi_1^2$. This model has exact bright soliton solutions for $s_2=s_1=+1$ and $\alpha>0$ [13]

$$\pm \tau = \tanh^{-1}(\rho) + 2\bar{\sigma} \tan^{-1}(2\bar{\sigma}\rho), \qquad (11)$$

where $\rho^2=(a_1^2-\phi_1^2)/(a_1^2+4\bar{\sigma}^2\phi_1^2)$, $a_1^2=2/\gamma$ being the maximum intensity of the FW. Exact dark soliton solutions exist for $s_2=-s_1=+1$ with nontrivial phase profiles [13].

For $|\alpha|\ll 1$ the nonlocality is strong, $\bar{\sigma}\gg 1$, and we can expand the response function $R(\tau)$ in Eq. (4). For bright solitons we then obtain the linear equation for the FW

$$s_1\partial_\tau^2\phi_1 - \phi_1 + \gamma P_1 R(\tau)\phi_1 = 0, \qquad (12)$$

where $\phi_2=\gamma P_1 R(\tau)$. In this eigenvalue problem the FW power $P_1=\int_{-\infty}^{\infty} \phi_1^2(\tau)d\tau$ plays the role of the eigenvalue, and bright solitons correspond to the fundamental mode of the waveguide structure created by the exponential response function. For $s_2=s_1=+1$ and $\alpha > 0$ Eq. (12) has exact bright soliton solutions in the form of the Bessel function of the first kind of order $2\bar{\sigma}$ [21]

$$\phi_1(\tau) = A_1 J_{2\bar{\sigma}}[\bar{\sigma}^2 \sqrt{2P_1 R(\tau)}]. \qquad (13)$$

For the single-soliton ground-state solution P_1 is found as the first zero of the derivative, $J_{2\bar{\sigma}}'(\sqrt{\bar{\sigma}^3 P_1})=0$, which assures that $\phi_1'(0)=0$. The amplitude A_1 is then found from the definition of P_1, giving $A_1^2\approx P_1/2 - [P_1/(\pi^2\bar{\sigma})]^{1/2}$.

In Fig. 1 we show the full width at half maximum of the FW intensity ϕ_1^2 of bright quadratic solitons versus the phase-mismatch parameter α. The analytical solutions obtained using the nonlocal analogy correctly captures the increase of the soliton width with decreasing α. The nonlocal model elegantly explains this effect: Because of the convolution in the nonlinearity in Eq. (9), representing a trapping potential or waveguide structure, this potential is always broader than the FW intensity profile itself, leading to its weaker confinement and larger width when the degree of nonlocality increases. The profiles shown in Fig. 1 further illustrates the excellent agreement of the numerical results and approximate nonlocal analytical solutions in both the weakly ($\alpha \gg 1$) and strongly ($\alpha \ll 1$) nonlocal limit.
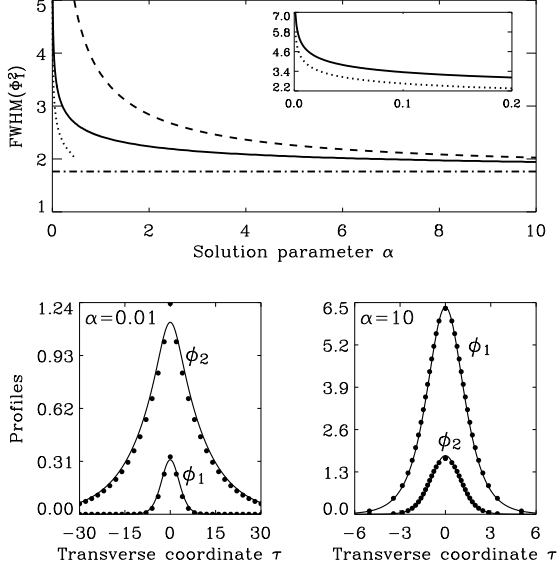
FIG. 1: Top: Numerically found FWHM(ϕ_1^2) of bright quadratic solitons versus α (solid), and the weakly nonlocal (dashed), strongly nonlocal (dotted), and cascading limit (chain-dashed) predictions. Bottom: Numerically found profiles (solid) and strongly nonlocal (left: $\alpha=0.01$) and weakly nonlocal (right: $\alpha=10$) solutions (dots). $s_2=s_1=+1$.
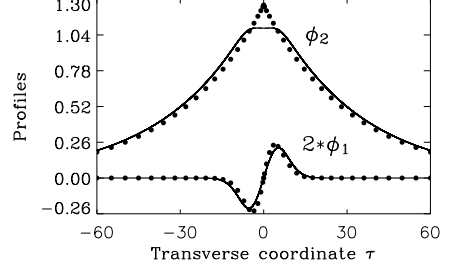
FIG. 2: Numerically found bound state of two out-of-phase bright solitons for $\alpha=0.001$ (solid) and the predicted strongly nonlocal solution (14) (dashed). $s_2=s_1=+1$.

The linear Eq. (12) describing the strongly nonlocal limit further predicts the existence of multi-hump bright solitons. Choosing P_1 as the N'th zero of the derivative, i.e., the N'th root in the equation $J'_{2\bar\sigma}(\sqrt{\bar\sigma^3 P_1})=0$, gives solitons with an odd number of humps $(2N-1)$, as discussed in [6]. However, this does not exhaust all soliton solutions supported by the model (12). There also exist antisymmetric solitons with an even number of intensity peaks. If the power P_1 is found as the N'th zero of the Bessel function itself, and not its derivative, i.e., as the N'th root in the equation $J_{2\bar\sigma}(\sqrt{\bar\sigma^3 P_1})=0$ (so that $\phi_1(0)=0$), then antisymmetric solitons with an even number of intensity peaks $(2N)$ exist with the form

$$\phi_1(\tau) = s_\tau A_1 J_{2\bar\sigma}[\bar\sigma^2 \sqrt{2P_1 R(\tau)}\,], \qquad (14)$$

where $s_\tau=\text{sign}(\tau)$. When P_1, e.g., is fixed by the first zero of $J_{2\bar\sigma}$ then the solution (14) is a two-peak antisymmetric soliton, which can be interpreted as a bound state of two out-of-phase fundamental solitons. In Fig. 2 we have shown the bound state of two out-of-phase fundamental solitons predicted by Eq. (14), and the corresponding numerically found solution for $\alpha=0.001$. We see again that the strongly nonlocal model provides an excellent prediction of this novel bound state quadratic soliton solution.

In fact, all higher order solitons can be thought of as a bound state of a number of individual solitons. Formation of such bound states follows naturally from the nonlocal character of the nonlinear interaction. Consider two out-of-phase solitons, for which the intensity in the overlapping region is always zero. In local Kerr media the nonlinear change in the refractive index is decreased in the overlap region, as compared to the index change generated by a single soliton. This leads to a mutual repulsion of the solitons. The nonlocality tends to increase the nonlinear change of the refractive index in the overlapping region, and for a sufficiently high degree of nonlocality, the index change may even be higher than for a soliton in isolation, despite the solitons being out-of-phase. This creates an attractive force and leads to formation of the bound state.

In the strongly nonlocal limit the bright soliton is the *fundamental mode* of the waveguide structure $R(\tau)$ and much narrower than the waveguide. In contrast the dark soliton is a *first mode* of the waveguide $R(\tau)$ *at the cut-off* and as such its width is comparable with that of the waveguide. Hence the expansion procedure leading to Eq. (12) is no longer justified. One can use it however, but for illustrative purposes only, to show what type of dark soliton solutions can be expected in the strongly nonlocal system. The linear equation for dark solitons in the strongly nonlocal limit will then have the form

$$s_1\partial_\tau^2\phi_1 - \phi_1 + \gamma[A_1^2 - Q_1 R(\tau)]\phi_1 = 0, \quad (15)$$

where $Q_1=\int_{-\infty}^{\infty}[A_1^2 - \phi_1^2(\tau)]d\tau$ is the complementary FW power and $\phi_2=\gamma[A_1^2 - Q_1 R(\tau)]$. For $s_2=-s_1=+1$ and $\alpha > 0$ Eq. (15) has exact dark soliton solutions in the form of the zero'th order Bessel function

$$\phi_1(\tau) = s_\tau \sqrt{2\alpha} J_0[\bar\sigma^2 \sqrt{2Q_1 R(\tau)}\,]. \qquad (16)$$

For the fundamental single-soliton solution Q_1 is found as the first zero $J_0[\sqrt{\bar\sigma^3 Q_1}]=0$, which gives $Q_1 = 5.8/\bar\sigma^3$ and assures that $\phi_1(0)=0$. As for bright solitons, choosing the N'th root gives novel multihump dark solitons with $2N-1$ dips in the intensity profile. However, the background amplitude $A_1 = 2\alpha$ is fixed and does not satisfy the self-consistency relation $Q_1=\int_{-\infty}^{\infty}[A_1^2 - \phi_1^2(\tau)]d\tau$.

In Fig. 3 we show the full width at half maximum of the FW intensity ϕ_1^2 of dark quadratic solitons versus the mismatch parameter α. The dark solitons have the constant background $\phi_1^2(\pm\infty) = 2\alpha$, $\phi_2(\pm\infty) = 1$. The analytical weakly nonlocal dark soliton solution exist for
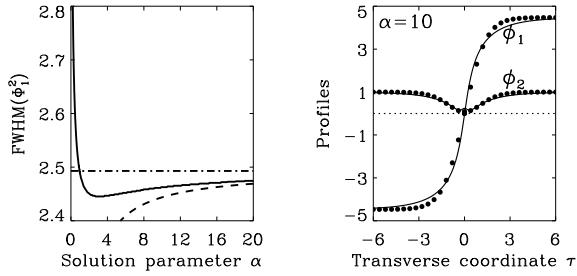
FIG. 3: Left: Numerically found FWHM(ϕ_1^2) of dark solitons versus α (solid) and the weakly nonlocal (dashed) and cascading limit (chain-dashed) predictions. Right: Numerically found profiles (dots) and weakly nonlocal solutions (solid) for $\alpha=10$. $\phi_1^2(\pm\infty)=2\alpha$, $\phi_2(\pm\infty)=1$, and $s_2=-s_1=+1$.

$\alpha > 2$ and was taken from Ref. [13]. Unlike bright solitons, whose width is a monotonic function of α, dark solitons are seen to have a clear minimum width of the FW at $\alpha=\alpha_0 \approx 3.1$.

Figure 3 confirms that the weakly nonlocal model correctly predicts how the soliton width decreases for $\alpha > \alpha_0$ when the mismatch parameter decreases. This, as well as the appearance of the minimum in the soliton width, is again elegantly explained by the nonlocal analogy: Because of the convolution in the nonlinearity in Eq. (9), representing the trapping potential or waveguide structure, the contribution from the constant background tends to contract this potential. This leads to a stronger confinement and thus a smaller width of the soliton. However, this is only true as long as the amplitude of the trapping potential is not affected by nonlocality, as in the weakly nonlocal regime. For a high degree of nonlocality (i.e., smaller value of α) not only the width of the trapping potential, but also its amplitude is affected. In this regime the nonlocality leads to a drop in the amplitude of the potential, resulting in a weaker confinement and an increase of the soliton width. The profiles shown in Fig. 3 further illustrates the excellent agreement of the numerical results and approximate nonlocal analytical solutions in the weakly nonlocal limit $\alpha \gg 1$.

In conclusion, we have used the analogy between parametric interaction in quadratic media and nonlocal Kerr-type nonlinearities to provide a physically intuitive theory for quadratic solitons, which allows for a deeper physical insight into their properties and an exact description of their profiles. It is well-known that quadratic solitons may be characterized by a single solution parameter α, which is an effective mismatch parameter depending on both the real phase-mismatch and the power. We have shown that the nonlocal theory provides a simple and elegant explanation for how the soliton width depends on the mismatch α.

In particular the nonlocal analogy provides simple physical models in both the large mismatch limit $\alpha \gg 1$ and the near cut-off limit $|\alpha| \ll 1$, corresponding to the regimes of weak and strong nonlocality, respectively. Our results show that the weakly nonlocal approximation gives an accurate description of quadratic solitons in a relatively broad range of their existence domain. Also, the simple linear physics of the strongly nonlocal limit has enabled us to find novel bound states of bright quadratic solitons and explain their formation using the natural concept of the nonlocality-based attraction between out-of-phase constituent solitons.

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